Model Theory – Lecture 10 – Prime models and categoricity 2

INFO

Next lecture we have two meetings (probably on zoom) to "discuss"

the project and pose some questions (again tuesday / thursday)

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TODAY

Last on previous lecture   Atomic model $\Rightarrow$ prime model

We want to show the contrary when the language (?) is countable

Afterwards – when does a theory have prime models?

– characterize $\aleph_0$ categorical theories

↱ here we use countability of $\mathcal{L}$

<u>Proof of (C7)</u> We apply LSD and get a countable prime model $\mathcal{M}$

We want to show atomicity for $\mathcal{M}$. Let Σ be any complete n-type. Notice, if $\Sigma = [\psi]$ for some formula ψ, then by the OTT it can be omitted in some model $\mathcal{N}$. Since $\mathcal{M}$ is prime, we have $\mathcal{M} \hookrightarrow \mathcal{N}$ elementarily, hence $\mathcal{M}$ has to omit Σ

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Theorem the following are equivalent

1) T has a prime model,

2) T has an atomic and countable model,

3) Isolated types are dense in the space of n-types (for every n)

Corollary: If the space of types of the theory is countable, then there must be a prime model

Proof (Corollary) In a compact Hausdorff countable space, isolated points must be dense

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Proof (Theorem) $(1 \iff 2)$ We already proved this

$(1 \Rightarrow 3)$ Let ψ be a coherent formula, then there exists a formula φ that has an isolated type and implies ψ. Indeed, since ψ is coherent, there is a model $\mathcal{N}$ and $\bar{a} \in |\mathcal{N}|$ such that

complete formula

$\mathcal{N} \models \psi(\bar{a})$, and, by primality of $\mathcal{M}$, $\mathcal{M} \hookrightarrow \mathcal{N}$ is elementary and we can extract $c \in |\mathcal{M}|$ such that $\mathcal{M} \models \varphi(\bar{c})$

$(3 \Rightarrow 1)$ Consider

$$\Sigma := \{ \neg\varphi \mid \varphi \text{ is a complete formula} \}$$

Notice Σ is not finitely supported (if φ exists, $\neg\varphi \in \Sigma$) Then, let $\mathcal{M}$ omit Σ by OTT. Then $\mathcal{M}$ is atomic / prime ∎

Theorem A The following statements are equivalent

1) Every type is isolated,

2) the space of type is finite

Theorem B The following statements are equivalent

↳ Assume $\mathcal{L}$ countable, φ has an infinite model

1) φ is $\aleph_0$-categorical;

2) the space of types is finite,

3) every model is atomic

Proof A) $1 \Rightarrow 2$) $tp^n(x) = \bigcup_{p \text{ isolate}} \{p\}$ is an open cover + the space is compact.

$2 \Rightarrow 1$) Here we use $\underset{+\text{finite}}{\text{Hausdorff}}$ and instantly conclude

Proof B) $3 \Rightarrow 1$) Since φ has a model, by LSD it is assumed countable. By assumption, all the countable models are atomic and they are isomorphic for a previous theorem (last theorem of last lecture)

$2 \Rightarrow 3$) A model can only realize isolated types (for theorem A), therefore it is prime

$1 \Rightarrow 2$) Take any non-isolated type. Consider M that realizes it. M can be chosen to be countable, and by assumption it is isomorphic to any other. On the other hand, we can omit that type in an isomorphic model. ↯

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